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A Computational Procedure for Calculating Matrix Elements and Clebsch–Gordan Coefficients of Simple Lie Algebras

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An algorithm is presented for evaluating matrix elements and Clebsch–Gordan coefficients of simple Lie algebras. The method is based upon simple properties of Lie algebras and representation theory and it is applicable to any classical or exceptional Lie algebra.

1. INTRODUCTION

Advances in the gauge theory of weak, electromagnetic and strong interactions have introduced high rank compact simple or semi-simple Lie groups into the Grand Unified theories [1–6]. Although the general properties of these groups are well understood, there is always a considerable computational problem in constructing explicit matrix realizations of irreducible representations (IRs) and evaluating Clebsch–Gordan coefficients (CGCs).

In this paper we shall develop a general analytic method for obtaining matrix elements of IRs and CGCs of simple Lie groups. The method is based upon simple properties of Lie algebras and representation theory, such as roots and weights, and it can be implemented by a computer program.

To demonstrate the practicability of the method, we shall analyze, in full, an example of a rank two Lie algebra. The generalization to higher rank Lie algebras will be discussed in the last section.

2. BASIC CONCEPTS

A general element of the Cartan subalgebra $\mathscr{H}$ of $\mathscr{L}$ will be denoted by h, α denotes a root and e_α is the corresponding basis element of $\mathscr{L}$ such that

$$[h, e_\alpha] = \alpha(h)\, e_\alpha. \tag{1}$$

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It is possible to associate with each root α a unique element h_α of $\mathscr{H}$ by the definition

$$B(h_\alpha, \mathscr{H}) = \alpha(h) \tag{2}$$

for all $h \in \mathscr{H}$, where B is the Killing form of $\mathscr{L}$ [7]. The elements h_α give rise to the Weyl canonical base with the usual commutation relations [8].

For each pair of roots α and $-\alpha$, there is a three-dimensional simple subalgebra of $\mathscr{L}$ [8] which can be constructed in the following way. Define $H_\alpha(\varepsilon\mathscr{H})$ by

$$H_\alpha = \{2/\langle\alpha, \alpha\rangle\}\, h_\alpha, \tag{3}$$

where $\langle\alpha, \alpha\rangle$ is defined by the Killing form of $\mathscr{L}$, as $\langle\alpha, \alpha\rangle = B(h_\alpha, h_\alpha) = \alpha(h_\alpha)$. Let E_α, $E_{-\alpha}$, be elements of the root subspaces [8] $\mathscr{L}_\alpha$ and $\mathscr{L}_{-\alpha}$, such that

$$B(E_\alpha, E_{-\alpha}) = 2/\langle\alpha, \alpha\rangle. \tag{4}$$

Then, for the pair $(\alpha, -\alpha)$ the subalgebra is generated by the following commutation relations:

$$\begin{aligned} [H_\alpha, E_\alpha] &= 2E_\alpha \\ [H_\alpha, E_{-\alpha}] &= -2E_{-\alpha} \\ [E_\alpha, E_{-\alpha}] &= H_\alpha. \end{aligned} \tag{5}$$

As $B(e_\alpha, e_{-\alpha}) = -1$ the basis elements e_α, $e_{-\alpha}$ of $\mathscr{L}$ can be expressed in terms of E_α and $E_{-\alpha}$ as follows:

$$\begin{aligned} E_\alpha &= \{2/\langle\alpha, \alpha\rangle\}^{1/2}\, e_\alpha \\ E_{-\alpha} &= -\{2/\langle\alpha, \alpha\rangle\}^{1/2}\, e_{-\alpha}. \end{aligned} \tag{6}$$

With the identification $J_3 = \frac{1}{2}H_\alpha$, $J_+ = E_\alpha$ and $J_- = E_{-\alpha}$ the operators J_3, J_+, J_- are the familiar angular momentum operators. Consequently, the above defined H_α, E_α, $E_{-\alpha}$ basis has the advantage of carrying the properties of the angular momentum algebra to any Lie algebra $\mathscr{L}$.

In the $SU(2)$ theory we know that every representation $D(j)$ is specified by the eigenvalue j ($j = 0, \frac{1}{2}, 1, \frac{3}{2},\ldots$), and each state of a multiplet is characterized by the eigenvalue m, taking integral or half-integral values in the range $-j \leqslant m \leqslant +j$, *ie* $2j + 1$ values in all. Moreover, each state of the multiplet is obtained from the state ψ_{jm} by the application of the raising and lowering operators which, with the phase convention of Condon and Shortley [9], are given by

$$J_\pm \psi_{jm} = \{(j \mp m)(j \pm m + 1)\}^{1/2}\, \psi_{jm\pm 1}. \tag{7}$$

It is then always possible to classify the basis functions of an IR of an arbitrary Lie algebra $\mathscr{L}$ according to its A_1-multiplets, and, using Eq. (7), to construct an explicit matrix realization of the IR.

The elements h_α of $\mathscr{H}$ may be represented by diagonal Hermitian matrices, while for each pair α and $-\alpha$ of roots the matrices e_α and $e_{-\alpha}$ may be chosen so that

$$\mathbf{e}_{-\alpha} = -\mathbf{e}_\alpha^\dagger \tag{8}$$

and correspondingly

$$\mathbf{E}_{-\alpha} = \mathbf{E}_\alpha^\dagger. \tag{9}$$

A matrix representation of a semi-simple Lie algebra can be completely specified if we know the matrices representing the elements of the Cartan subalgebra $\mathscr{H}_\alpha$, and the matrices representing the elements E_α, $E_{-\alpha}$ of $\mathscr{L}_\alpha$, $\mathscr{L}_{-\alpha}$ respectively, for every simple root α. Then, from the commutation relations

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta} \tag{10}$$

all the other matrices can be constructed.

3. Explicit Matrix Realization of an IR of a Simple Lie Algebra

If an IR Γ of a simple Lie algebra $\mathscr{L}$ has weight multiplicity not exceeding one, then the action of the raising and lowering operators on the basis functions of Γ will trivially generate the matrix elements of this representation. However, if the weight multiplicity is greater than one, considerable complications arise due to the fact that more than one basis function corresponds to the same weight.

There is a straightforward method for the construction of diagonal matrices representing H_α, when the weight system is known.

We can define the weights λ of an IR as the eigenvalues of an operator $W(h)$, $h \in \mathscr{H}$, on the basis functions $\psi_1, \psi_2, \ldots, \psi_l$, where l is the rank of the algebra. This implies that the diagonal matrices are given by

$$\Gamma(h)_{jj} = \lambda_j(h), \qquad h \in \mathscr{H},\ j = 1, 2, \ldots, l. \tag{11}$$

In the basis defined by Eqs. (5), we have

$$(\Gamma(H_\alpha))_{jj} = \lambda_j(H_\alpha) = \{2/\langle\alpha, \alpha\rangle\}\, \lambda_j(h_\alpha). \tag{12}$$

Defining the linear functional $\lambda_j : \mathscr{H} \to \mathbb{C}$ [8] by

$$\lambda_j(h_\alpha) = \langle\lambda_j, \alpha\rangle \tag{13}$$

Eq. (12) becomes

$$(\Gamma(H_\alpha))_{jj} = \lambda_j(H_\alpha) = 2\langle\lambda_j, \alpha\rangle/\langle\alpha, \alpha\rangle. \tag{14}$$

To determine the $\Gamma(E_\alpha)$, a simple observation considerably simplifies the calculations.

Equation (1) can be written

$$[\boldsymbol{\Gamma}(h), \boldsymbol{\Gamma}(e_\alpha)] = \alpha(h)\, \boldsymbol{\Gamma}(e_\alpha). \tag{15}$$

In particular, the pq element is

$$\{\Gamma(h)_{pp} - \Gamma(h)_{qq} - \alpha(h)\}(\Gamma(e_\alpha))_{pq} = 0. \tag{16}$$

Equation (16) tells us that $(\Gamma(e_\alpha))_{pq} \neq 0$ only if

$$\Gamma(h)_{pp} - \Gamma(h)_{qq} = \alpha(h) \tag{17}$$

i.e., $(\Gamma(e_\alpha))_{pq} \neq 0$ if the difference between the pth weight and the qth weight is $\alpha(h)$.

Now let us suppose that we have constructed the matrices representing the generators H_α, α simple root, using Eq. (16). Then we can partition these diagonal matrices into blocks according to their A_1-subalgebra content. If all the weights are simple, then there is a unique block form for each A_1-subalgebra. If some of the weights have multiplicity greater than one, there is an ambiguity in deciding which element of the $\Gamma(H_\alpha)$ belongs to each A_1-subalgebra IR. In the following example we shall show how one can remove this ambiguity.

The 27-dimensional representation of G_2 has some of its weights with multiplicity greater than one. In Fig. 1a we have enumerated the weights according to their lexicographical order. The vertical lines join pairs of eigenvectors ψ_λ and ψ'_λ, such that $(\psi_\lambda, E_{-\alpha_2}\psi'_\lambda) \neq 0$, while the horizontal lines join pairs ψ_λ, ψ'_λ of eigenvectors such that $(\psi_\lambda, E_{-\alpha_1}\psi'_\lambda) \neq 0$. With the following values of the normalization of the quantities $\langle \alpha_i, \alpha_i \rangle$, $i = 1, 2$,

$$\langle \alpha_1, \alpha_1 \rangle = \tfrac{1}{4}, \qquad \langle \alpha_2, \alpha_2 \rangle = \tfrac{1}{12} \tag{18}$$

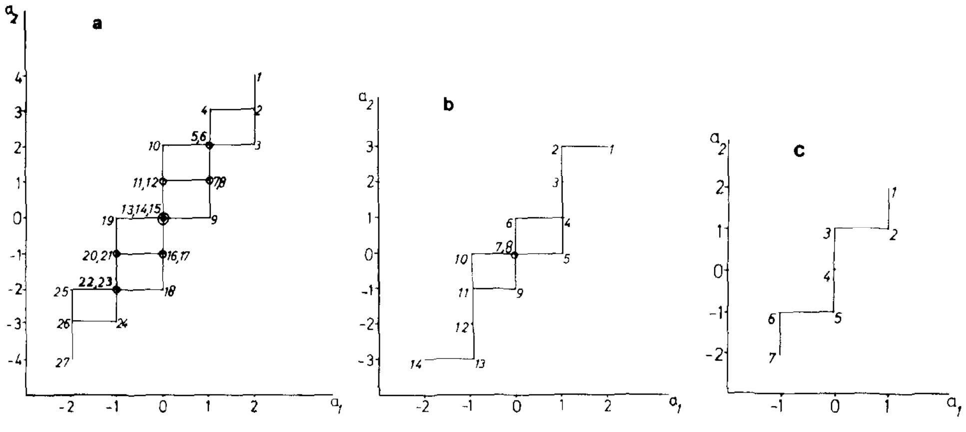

FIG. 1. The weight diagrams of (a) 27-dimensional representation, (b) 14-dimensional representation, (c) 7-dimensional representation. The circles represent weight multiplicities.

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and the use of the relation $J_3 = \frac{1}{2}H_\alpha$, Eq. (14) gives for the **27** representation

$$\begin{aligned}\Gamma^{27}(H_{\alpha_1}) &= \tfrac{1}{4}\,\mathrm{diag}(0, \tfrac{1}{2}, 1, -\tfrac{1}{2}, 0, 0, \tfrac{1}{2}, \tfrac{1}{2}, 1, -1\\ &\quad -1, -\tfrac{1}{2}, -\tfrac{1}{2}, 0, 0, 0, \tfrac{1}{2}, \tfrac{1}{2}, -1, -1, -\tfrac{1}{2}, 0, 0, \tfrac{1}{2}, -1, -\tfrac{1}{2}, 0)\\ \Gamma^{27}(H_{\alpha_2}) &= \tfrac{1}{4}\,\mathrm{diag}(1, 0, -1, \tfrac{3}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{3}{2}, 2\\ &\quad 1, 1, 0, 0, 0, -1, -1, -2, \tfrac{3}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{3}{2}, 1, 0, -1).\end{aligned} \tag{19}$$

Using Eq. (17), the matrix $\Gamma^{27}(H_{\alpha_2})$, for example, can be written in an A_1-block form (we omit the constant factor $\frac{1}{4}$).

$$\Gamma^{27}(H_{\alpha_2}) = \mathrm{diag}(\underbrace{1,\ 0,\ -1}_{A_1\text{-triplet}};\ \underbrace{\tfrac{3}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ -\tfrac{1}{2},\ -\tfrac{1}{2},\ -\tfrac{3}{2}}_{A_1\text{-tetraplet},\ A_1\text{-doublet}};$$

$$\underbrace{2,\ 1,\ 1,\ 0,\ 0,\ 0,\ -1,\ -1,\ 2}_{A_1\text{-pentaplet},\ A_1\text{-triplet},\ A_1\text{-singlet}}; \tag{20}$$

$$\underbrace{\tfrac{3}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ -\tfrac{1}{2},\ -\tfrac{1}{2},\ -\tfrac{3}{2}}_{A_1\text{-tetraplet},\ A_1\text{-doublet}};\ \underbrace{1,\ 0,\ -1}_{A_1\text{-triplet}}).$$

The lines in the above expression indicate the ambiguity of assigning the eigenvalues to a particular multiplet according to Eq. (17). This ambiguity will also be reflected in the matrix $\Gamma(E_{\alpha_2})$. In Fig. 2 we have represented the matrix $\Gamma(E_{\alpha_2})$ with its various A_1-blocks.

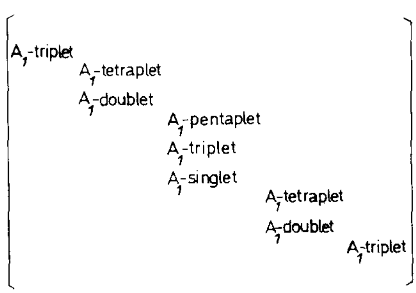

FIG. 2. The matrix $\Gamma^{27}(E_{\alpha_2})$ in its A_1 − subalgebra block form.

If we go through the same computational procedure for the other generator H_{α_1}, we shall end up with a matrix $\mathbf{\Gamma}(E_{\alpha_1})$ of a similar form as in Fig. 2 (with a different arrangement of the A_1-blocks). To remove the ambiguities present in these two matrices, we choose one direction in the weight space, let us say the α_2-direction, in which we fix the multiplets by an arbitrary choice of the states. This choice will fix unambiguously the matrix $\mathbf{\Gamma}(E_{\alpha_2})$. The matrix elements in the α_1-direction can then be computed using the commutation relations of the algebra. In this particular example we have chosen the states ψ_5, ψ_7 (see Fig. 1a) to belong to the tetraplet (ψ_4, ψ_5, ψ_7, ψ_9) while the states ψ_6, ψ_8 are chosen such as to form a α_2-doublet (ψ_6, ψ_8); the states ψ_{10}, ψ_{11}, ψ_{13}, ψ_{16}, ψ_{18} form a pentaplet; the states ψ_{12}, ψ_{14}, ψ_{17} form a triplet; the state ψ_{15} is a singlet; the states ψ_{19}, ψ_{20}, ψ_{22}, ψ_{24} form a tetraplet, and finally the states ψ_{21}, ψ_{23} form a doublet.

As $E_{\alpha_1}\psi_5$ and $E_{\alpha_1}\psi_6$ are both proportional to ψ_3 (Fig. 1a) let

$$\begin{aligned} E_{\alpha_1}\psi_5 &= a\psi_3 \\ E_{\alpha_1}\psi_6 &= b\psi_3. \end{aligned} \tag{21}$$

The matrices can be chosen to be real, so that relation (9) can be written

$$\mathbf{\Gamma}(E_\alpha) = \tilde{\mathbf{\Gamma}}(E_{-\alpha}). \tag{22}$$

Because of relation (21) the following results would be valid:

$$\begin{aligned} &\Gamma(E_{\alpha_1})_{35} = a, \qquad \Gamma(E_{\alpha_1})_{j5} = 0 \qquad j \neq 3 \\ &\Gamma(E_{\alpha_1})_{36} = b, \qquad \Gamma(E_{\alpha_1})_{j6} = 0 \qquad j \neq 3. \end{aligned} \tag{23}$$

There would be complex numbers λ and μ such that

$$E_{-\alpha_1}\psi_3 = \mu\psi_5 + \lambda\psi_6. \tag{24}$$

This implies that

$$\begin{aligned} &\Gamma(E_{-\alpha_1})_{53} = \mu, \qquad \Gamma(E_{-\alpha_1})_{j3} = 0 \qquad j \neq 5, 6 \\ &\Gamma(E_{-\alpha_1})_{63} = \lambda. \end{aligned} \tag{25}$$

Thus, relations (22), (23) and (25) give

$$a = \mu, \qquad b = \lambda. \tag{26}$$

Hence, relation (24) becomes

$$E_{-\alpha_1}\psi_3 = a\psi_5 + b\psi_6. \tag{27}$$

The action of E_{α_2} on relation (27), as $[E_{-\alpha_1}, E_{\alpha_2}] = 0$, will give

$$\begin{aligned} E_{\alpha_2}E_{-\alpha_1}\psi_3 &= aE_{\alpha_2}\psi_5 + bE_{\alpha_2}\psi_6 \\ &= \sqrt{3}a\psi_4 \qquad \text{(from our choice in the } \alpha_2\text{-direction and Eq. (7))} \\ E_{-\alpha_1}E_{\alpha_2}\psi_3 &= E_{-\alpha_1}(\sqrt{2}\psi_2) \\ &= \sqrt{2}\psi_4 \qquad \text{(because } E_{-\alpha_1}\psi_2 = \psi_4 \text{ from Eq. (7)).} \end{aligned}$$

Thus

$$a = \sqrt{\tfrac{2}{3}}. \tag{28}$$

TABLE I

Matrix Elements of the 27, 14 and 7 Representations of G_2 (Only the Non-zero Values v of the (i, j) Coordinates Are Given, in an Array (i, j, v))

27 Representation		14 Representation		7 Representation	
$\Gamma(E_{\alpha_1})$	$\Gamma(E_{\alpha_2})$	$\Gamma(E_{\alpha_1})$	$\Gamma(E_{\alpha_2})$	$\Gamma(E_{\alpha_1})$	$\Gamma(E_{\alpha_2})$
2 4 1	1 2 $\sqrt{2}$	1 2 1	2 3 $\sqrt{3}$	2 3 1	1 2 1
3 5 $\sqrt{2}/\sqrt{3}$	2 3 $\sqrt{2}$	4 6 1	3 4 2	5 6 1	3 4 $1/\sqrt{2}$
5 10 $\sqrt{2}/\sqrt{3}$	4 5 $\sqrt{3}$	5 7 $\sqrt{3}/\sqrt{2}$	4 5 $\sqrt{3}$		4 5 $1/\sqrt{2}$
3 6 $2/\sqrt{3}$	5 7 2	7 10 $\sqrt{3}/\sqrt{2}$	6 7 $\sqrt{2}$		6 7 1
6 10 $2/\sqrt{3}$	6 8 1	5 8 $1/\sqrt{2}$	7 9 $\sqrt{2}$		
7 11 $\sqrt{2}/\sqrt{3}$	7 9 $\sqrt{3}$	8 10 $1/\sqrt{2}$	10 11 $\sqrt{3}$		
7 12 $1/\sqrt{3}$	10 11 2	9 11 1	11 12 2		
8 11 $1/\sqrt{3}$	11 13 $\sqrt{6}$	13 14 1	12 13 $\sqrt{3}$		
8 12 $-\sqrt{2}/\sqrt{3}$	12 14 $\sqrt{2}$				
9 13 $1/\sqrt{3}$	13 16 $\sqrt{6}$				
13 19 $1/\sqrt{3}$	14 17 $\sqrt{2}$				
9 14 $1/\sqrt{2}$	16 19 2				
14 19 $1/\sqrt{2}$	19 20 $\sqrt{3}$				
9 15 $\sqrt{8}/\sqrt{7}$	20 22 2				
15 19 $\sqrt{8}/\sqrt{7}$	21 23 1				
16 20 $\sqrt{2}/\sqrt{3}$	22 24 $\sqrt{3}$				
16 21 $1/\sqrt{3}$	25 26 1				
17 20 $1/\sqrt{3}$					
17 21 $-\sqrt{2}/\sqrt{3}$					
18 22 $\sqrt{2}/\sqrt{3}$					
22 25 $\sqrt{2}/\sqrt{3}$					
18 23 $2/\sqrt{3}$					
23 25 $2/\sqrt{3}$					
24 26 1					

From $[E_{\alpha_1}, E_{-\alpha_1}] = H_{\alpha_1}$ we have when applied to ψ_3,

$$E_{\alpha_1} E_{-\alpha_1} \psi_3 - E_{-\alpha_1} E_{\alpha_1} \psi_3 = H_{\alpha_1} \psi_3$$

$$E_{\alpha_1}(a\psi_5 + b\psi_6) = 2\,\frac{\langle 2\alpha_1 + 2\alpha_2, \alpha_1\rangle}{\langle \alpha_1, \alpha_1\rangle}\,\psi_3$$

$$(a^2 + b^2)\,\psi_3 = \left(2\,\frac{2\langle \alpha_1, \alpha_1\rangle}{\langle \alpha_1, \alpha_1\rangle} + 2\,\frac{2\langle \alpha_2, \alpha_1\rangle}{\langle \alpha_1, \alpha_1\rangle}\right)\psi_3$$

$$(a^2 + b^2)\,\psi_3 = (4 + 2(-1))\,\psi_3$$ (from the Cartan matrix of G_2 we have $A_{11} = 2$, $A_{21} = -1$)

$$a^2 + b^2 = 2$$

and from Eq. (28), we get $b = 2/\sqrt{3}$.

The above calculations fix the matrix elements around the (2, 3, 5, 4) loop. All the other loops of the weight diagram of Fig. 1a can be calculated in the same way. This results in a matrix representation of E_{α_1}.

Table I gives the non-zero matrix elements of the generators $\Gamma(E_{\alpha_1})$ and $\Gamma(E_{\alpha_2})$ of the 27-dimensional representation of G_2. We shall discuss later the generalization of the above computational procedure for large l.

4. A Procedure for Generating CGCs

An important by-product of the method developed in Section 3 is a procedure for evaluating CGCs. To establish our notation, we shall denote the basis functions of two IRs of a simple compact Lie group by $\psi_\mu^{(a)}$ and $\psi_\nu^{(b)}$, where a and b stand for the dimensions of the IRs and μ, ν for their weights in lexicographical order. Then, the CGCs will be given by [10]

$$\psi_m^{(j\gamma)} = \sum_{\mu\nu} \begin{pmatrix} a & b & \big| & j\gamma \\ \mu & \nu & \big| & m \end{pmatrix} \psi_\mu^{(a)} \otimes \psi_\nu^{(b)}, \tag{29}$$

where the index γ distinguishes those IRs which appear in a Clebsch–Gordan Series (CGS) more than once. We can also express the product basis functions as linear combinations of the basis functions of the IRs, i.e.,

$$\psi_\mu^{(a)} \otimes \psi_\nu^{(b)} = \sum_{j\gamma m} \begin{pmatrix} j\gamma & \big| & a & b \\ m & \big| & \mu & \nu \end{pmatrix} \psi_m^{(j\gamma)}. \tag{30}$$

The knowledge of the matrix elements and the use of the raising and lowering operators is sufficient to determine the CGCs of a given CGS. An example will elucidate our discussion.

Let us suppose we want to evaluate the CGCs of the CGS

$$7 \otimes 7 = 27 \oplus 14 \oplus 7 \oplus 1 \tag{31}$$

of G_2. Using the methods of Section 3 we construct first a matrix realization of the IRs appearing in relation (31). The results of the computation are in Table I.

We shall denote the basis functions of the representations in relation (31) by $\psi^{7}_{i,\nu}$ $(i = 1, 2,..., 7)$, $\psi^{7'}_{j,\mu}$ $(j = 1, 2,..., 7)$, $\psi^{27}_{k,\pi}$ $(k = 1, 2,..., 27)$, $\psi^{14}_{l,\lambda}$ $(l = 1, 2,..., 14)$, $\psi^{7}_{m,\omega}$ $(m = 1, 2,..., 7)$ and $\psi^{1}_{1,(0,0)}$. The Greek indices indicate the weight systems, while the Latin indices i, j, k, l, m specify the position of a state in the weight diagrams (Fig. 1). In terms of the basis functions relation (31) can be written

$$\begin{aligned}\psi^{7}_{i,\nu} \otimes \psi^{7'}_{j,\mu} = &\left(\begin{array}{c|cc} 27 & 7 & 7' \\ k,\pi & i,\nu & j,\mu \end{array}\right) \psi^{27}_{k,\pi} + \left(\begin{array}{c|cc} 14 & 7 & 7' \\ l,\lambda & i,\nu & j,\mu \end{array}\right) \psi^{14}_{l,\lambda} \\ &+ \left(\begin{array}{c|cc} 7 & 7 & 7' \\ m,\omega & i,\nu & j,\mu \end{array}\right] \psi^{7}_{m,\omega} + \left(\begin{array}{c|cc} 1 & 7 & 7' \\ 1,(0,0) & i,\nu & j,\mu \end{array}\right) \psi^{1}_{1,(0,0)}.\end{aligned} \tag{32}$$

To calculate the CGCs $\left(\begin{smallmatrix} 27 \\ k,\pi \end{smallmatrix} \middle| \begin{smallmatrix} 7 & 7' \\ i,\nu & j,\mu \end{smallmatrix}\right)$, we start with the highest weight of the **27** representation. In terms of the basis functions of the 7's we have

$$\psi^{27}_{1,(2,4)} = 1 \cdot \psi^{7}_{1,(1,2)} \otimes \psi^{7'}_{1,(1,2)}. \tag{33}$$

The successive application of the lowering and raising operators on the states of (33) will result in a complete determination of the $\left(\begin{smallmatrix} 27 \\ k,\pi \end{smallmatrix} \middle| \begin{smallmatrix} 7 & 7' \\ i,\nu & j,\mu \end{smallmatrix}\right)$ coefficients.

Applying this method, we must be careful when we encounter a state with multiplicity greater than one. In Fig. 3 we have isolated the first state with internal multiplicity two of the **27** representation. The application of $E_{-\alpha_1}$ to the state $\psi^{27}_{3,(2,2)}$ gives

$$E_{-\alpha_1} \psi^{27}_{3,(2,2)} = \sqrt{\frac{2}{3}}\, \psi^{27}_{5,(1,2)} + \frac{2}{\sqrt{3}}\, \psi^{27}_{6,(1,2)}. \tag{34}$$

The result of applying $E_{-\alpha_2}$ to $\psi^{27}_{2,(2,3)}$ is

$$\psi^{27}_{3,(2,2)} = \psi^{7}_{2,(1,1)} \otimes \psi^{7'}_{2,(1,1)}. \tag{35}$$

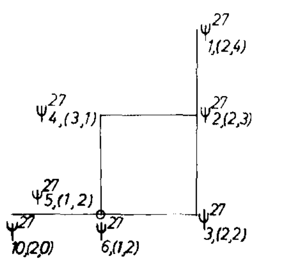

FIG. 3. The first state with multiplicity two of the 27 representation.

Then, the left-hand side of Eq. (34) becomes

$$E_{-\alpha_1}(\psi^7_{2,(1,1)} \otimes \psi^{7'}_{2,(1,1)}) = \psi^7_{3,(0,1)} \otimes \psi^{7'}_{2,(1,1)} + \psi^7_{2,(1,1)} \otimes \psi^{7'}_{3,(0,1)}. \tag{36}$$

Because we have chosen the state 5 to belong to an E_{α_2}-triplet (see Section 3), when $E_{-\alpha_2}$ is applied to the basis function $\psi^{27}_{4,(3,1)}$ we find

$$\begin{aligned}\psi^{27}_{5,(1,2)} = {} & \frac{1}{\sqrt{3}} \psi^7_{4,(0,0)} \otimes \psi^{7'}_{1,(1,2)} + \frac{1}{\sqrt{6}} \psi^7_{3,(0,1)} \otimes \psi^{7'}_{2,(1,1)} \\ & + \frac{1}{\sqrt{6}} \psi^7_{2,(1,1)} \otimes \psi^{7'}_{3,(0,1)} + \frac{1}{\sqrt{3}} \psi^7_{1,(1,2)} \otimes \psi^{7'}_{4,(0,0)}.\end{aligned} \tag{37}$$

If we substitute relations (37) and (36) into (34) we get a relation from which the state $\psi^{27}_{6,(1,2)}$ is determined in terms of its 7 and 7′ components. The result is

$$\begin{aligned}\psi^{27}_{6,(1,2)} = {} & \frac{1}{\sqrt{3}} \psi^7_{3,(0,1)} \otimes \psi^{7'}_{2,(1,1)} + \frac{1}{\sqrt{3}} \psi^7_{2,(1,1)} \otimes \psi^{7'}_{3,(0,1)} \\ & - \frac{1}{\sqrt{6}} \psi^7_{4,(0,0)} \otimes \psi^{7'}_{1,(1,2)} - \frac{1}{\sqrt{6}} \psi^7_{1,(1,2)} \otimes \psi^{7'}_{4,(0,0)}.\end{aligned} \tag{38}$$

To evaluate the CGCs $\left(\begin{smallmatrix}14 \\ l,\lambda\end{smallmatrix} \middle| \begin{smallmatrix}7 & 7' \\ i,\nu & j,\mu\end{smallmatrix}\right)$, we start again with the highest weight of the **14** representation (2, 3). However, there is another state with the same weight, belonging to the **27** representation, given by

$$\psi^{27}_{2,(2,3)} = \frac{1}{\sqrt{2}} (\psi^7_{2,(1,1)} \otimes \psi^{7'}_{1,(1,2)} + \psi^7_{1,(1,2)} \otimes \psi^{7'}_{2,(1,1)}).$$

To define the state $\psi^{14}_{1,(2,3)}$ we choose the orthogonal combination to the above state

$$\psi^{14}_{1,(2,3)} = \frac{1}{\sqrt{2}} (\psi^7_{2,(1,1)} \otimes \psi^{7'}_{1,(1,2)} - \psi^7_{1,(1,2)} \otimes \psi^{7'}_{2,(1,1)}), \tag{39}$$

and we repeat exactly the same procedure. With this method all the CGCs of relation (32) can be easily calculated.

The results of the full calculation are shown in Table II.

5. Generalizations

As we go to higher rank algebras and representations the weight system becomes more complicated and the difficulty of the method is to solve the labeling problem of the states which appear in the various loops. Towards this direction we have developed a computer program [11] for the physically interesting $SO(4n+2)$ groups

TABLE II

CGCs of the CGS $7 \otimes 7 = 27 \oplus 14 \oplus 7 \oplus 1$ of G_2

a. 27 Representation

State	7,7 States	CGCs	State	7,7 States	CGCs	State	7,7 States	CGCs
1	1 1	1	13	3 5	$1/\sqrt{6}$	22	4 7	$1/\sqrt{3}$
2	1 2	$1/\sqrt{2}$		4 4	$\sqrt{2}/\sqrt{3}$		5 6	$1/\sqrt{6}$
	2 1	$1/\sqrt{2}$		5 3	$1/\sqrt{6}$		6 5	$1/\sqrt{6}$
3	2 2	1	14	1 7	1/2		7 4	$1/\sqrt{3}$
4	1 3	$1/\sqrt{2}$		2 6	1/2	23	4 7	$-1/\sqrt{6}$
	3 1	$1/\sqrt{2}$		6 2	1/2		5 6	$1/\sqrt{3}$
5	1 4	$1/\sqrt{3}$		7 1	1/2		6 5	$1/\sqrt{3}$
	2 3	$1/\sqrt{6}$	15	1 7	$-1/3$		7 4	$-1/\sqrt{6}$
	3 2	$1/\sqrt{6}$		2 6	1/3	24	5 7	$1/\sqrt{2}$
	4 1	$1/\sqrt{3}$		3 5	7/16		7 5	$1/\sqrt{2}$
6	1 4	$-1/\sqrt{6}$		4 4	$-7/16$	25	6 6	1
	2 3	$1/\sqrt{3}$		5 3	7/16	26	6 7	$1/\sqrt{2}$
	3 2	$1/\sqrt{3}$		6 2	1/3		7 6	$1/\sqrt{2}$
	4 1	$-1/\sqrt{6}$		7 1	$-1/3$	27	7 7	1
7	1 5	$1/\sqrt{6}$	16	4 5	$1/\sqrt{2}$			
	2 4	$1/\sqrt{3}$		5 4	$1/\sqrt{2}$			
	4 2	$1/\sqrt{3}$	17	2 7	$1/\sqrt{2}$			
	5 1	$1/\sqrt{6}$		7 2	$1/\sqrt{2}$			
8	1 5	$-1/\sqrt{3}$	18	5 5	1			
	2 4	$-1/\sqrt{6}$	19	3 6	$1/\sqrt{2}$			
	4 2	$1/\sqrt{6}$		6 3	$1/\sqrt{2}$			
	5 1	$-1/\sqrt{3}$	20	3 7	$1/\sqrt{6}$			
9	2 5	$1/\sqrt{2}$		4 6	$1/\sqrt{3}$			
	5 2	$1/\sqrt{2}$		6 4	$1/\sqrt{3}$			
10	3 3	1		7 3	$1/\sqrt{6}$			
11	3 4	$1/\sqrt{2}$	21	3 7	$-1/\sqrt{3}$			
	4 3	$1/\sqrt{2}$		4 6	$1/\sqrt{6}$			
12	1 6	$1/\sqrt{2}$		6 4	$1/\sqrt{6}$			
	6 1	$1/\sqrt{2}$		7 3	$-1/\sqrt{3}$			

Table continued

TABLE II (continued)

b. 14 Representation

State	7,7 States	CGCs	State	7,7 States	CGCs	State	7,7 States	CGCs
1	1 2	$-1/\sqrt{2}$		4 3	$1/\sqrt{3}$	10	3 6	$-1/\sqrt{2}$
	2 1	$1/\sqrt{2}$		6 1	$1/\sqrt{6}$		6 3	$1/\sqrt{2}$
2	1 3	$-1/\sqrt{2}$	7	1 7	$-1/2\sqrt{3}$	11	3 7	$-1/\sqrt{6}$
	3 1	$1/\sqrt{2}$		2 6	$-1/2\sqrt{3}$		4 6	$-1/\sqrt{3}$
3	1 4	$-1/\sqrt{3}$		3 5	$-1/\sqrt{3}$		6 4	$1/\sqrt{3}$
	2 3	$-1/\sqrt{6}$		5 3	$1/\sqrt{3}$		7 3	$1/\sqrt{6}$
	3 2	$1/\sqrt{6}$		6 2	$1/2\sqrt{3}$	12	4 7	$-1/\sqrt{3}$
	4 1	$1/\sqrt{3}$		7 1	$1/2\sqrt{3}$		5 6	$-1/\sqrt{6}$
4	1 5	$-1/\sqrt{6}$	8	1 7	$1/2$		6 5	$1/\sqrt{6}$
	2 4	$-1/\sqrt{3}$		2 6	$-1/2$		7 4	$1/\sqrt{3}$
	4 2	$1/\sqrt{3}$		6 2	$1/2$	13	5 7	$-1/\sqrt{2}$
	5 1	$1/\sqrt{6}$		7 1	$-1/2$		7 5	$1/\sqrt{2}$
5	2 5	$-1/\sqrt{2}$	9	2 7	$-1/\sqrt{6}$	14	6 7	$-1/\sqrt{2}$
	5 2	$1/\sqrt{2}$		4 5	$-1/\sqrt{3}$		7 6	$1/\sqrt{2}$
6	1 6	$-1/\sqrt{6}$		5 4	$1/\sqrt{3}$			
	3 4	$-1/\sqrt{3}$		7 2	$1/\sqrt{6}$			

c. 7 Representation

State	7,7 States	CGCs	State	7,7 States	CGCs	State	7,7 States	CGCs
1	1 4	$-1/\sqrt{6}$		4 3	$-1/\sqrt{6}$		5 4	$-1/\sqrt{6}$
	2 3	$1/\sqrt{3}$		6 1	$1/\sqrt{3}$		7 2	$1/\sqrt{3}$
	3 2	$-1/\sqrt{3}$	4	1 7	$-1/\sqrt{6}$	6	3 7	$-1/\sqrt{3}$
	4 1	$1/\sqrt{6}$		2 7	$-1/\sqrt{6}$		4 6	$1/\sqrt{6}$
2	1 5	$-1/\sqrt{3}$		3 5	$1/\sqrt{6}$		6 4	$-1/\sqrt{6}$
	2 4	$1/\sqrt{6}$		5 3	$-1/\sqrt{6}$		7 3	$1/\sqrt{3}$
	4 2	$-1/\sqrt{6}$		6 2	$1/\sqrt{6}$	7	4 7	$-1/\sqrt{6}$
	5 1	$1/\sqrt{3}$		7 1	$-1/\sqrt{6}$		5 6	$1/\sqrt{3}$
3	1 6	$-1/\sqrt{3}$	5	2 7	$-1/\sqrt{3}$		6 5	$-1/\sqrt{3}$
	3 4	$1/\sqrt{6}$		4 5	$1/\sqrt{6}$		7 4	$1/\sqrt{6}$

d. 1 Representation

State	7,7 States	CGCs
1	1 7	$-1/2$
	2 6	$1/2$
	6 2	$-1/2$
	7 1	$1/2$

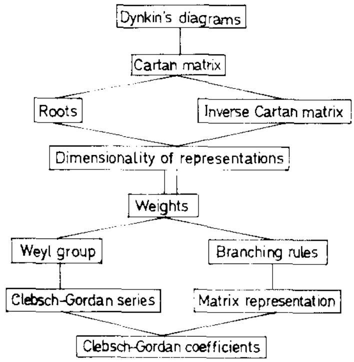

FIG. 4. A diagramatic representation of a Lie algebra.

[3, 5]. The program implements general algorithm for constructing matrix elements of IR of any semi-simple Lie algebra L of rank l consisting of the following steps:

(a) An arbitrary choice of the multiplets in the α_l-direction.

(b) For each of the other α_i-multiplets ($i = l-1, l-2, \dots, 2, 1$) we consider all the loops (α_i, α_j) with $i > j$ and $i, j = l, l-1, \dots, 2, 1$.

(c) If the loop (α_i, α_j) cannot specify the states of the α_j-multiplet, we make a choice of the states belonging to the α_j-multiplet.

Once the matrix elements have been evaluated using the above algorithm the program proceeds to the evaluation of the CGCs of a particular CGS. In an application of this program to the $SO(10)$ theory the matrix elements of the 126, 120, 16 and 10 IRs have been evaluated and the CGS of the CGS $16 \otimes 16 = 126 \oplus 120 \oplus 10$ have been calculated [11].

In the CGCs theory [12] there is a complication [13, 14] if an IR appears more than once in a CGS. In that case, the method of Section 4 is also applicable, the only difference being that the basis functions corresponding to the highest weight of the IR Γ^{α,γ_i} must be an orthogonal combination of the basis function corresponding to the highest weights of the representations Γ^{α,γ_j} with $j < i$.

An implementation of the above method to other Lie algebras is needed and this requires a systematic study of the labeling problem. We believe that it can be done in a systematic way using a computer [11]. This opens the possibility of developing a Lie algebra computer package program according to Fig. 4, where the only input would consist of the type and the rank of a particular Lie algebra.

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